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ON THE STABILITY OF STATIONARY MOTIONS OF NON-CONSERVATIVE MECHANICAL SYSTEMS*

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The problem of the stability of stationary motions (SM) of mechanical systems admitting of first integrals and a function that does not grow along the motions is considered. Theorems are proposed on the stability and asymptotic stability in parts of the variables, as well as on the instability of the SM of such systems. The general situations are illustrated with an example of the motion of a heavy inhomogeneous sphere over a plane with friction.

1. We consider a scleronomic mechanical system that admits of time-independent first integrals $U_1(x) = c_1, \ldots, U_k(x) = c_k$, and a time-independent function $U_0(x)$ that does not grow along the system motions.

We assume that the functions $U_0(x), U_1(x), \ldots, U_k(x)$ are continuously differentiable with respect to the variables $x = (x_1, \ldots, x_n)$ therein. All or certain generalized coordinates and velocities or momenta of the system, quasicoordinates, certain functions of these quantities etc., can be these variables.

Theorem 1. If a function $U_0(x)$ that does not grow along the system motions has a strict local minimum for constant values of the integrals $U_i(x) = c_i$ $(i = 1, \ldots, k)$ of this system, then the values of the variables making this function a minimum correspond to the stable real motion of the system (this motion is usually called stationary).

Theorem 2. If the stationary motion (SM) makes the function $U_0(x)$ a strict local minimum and is isolated for constant values of the integrals $U_i(x) = c_i$ $(i = 1, \ldots, k)$ of the motions along which the function $U_0(x)$ remains constant, then every perturbed motion that is sufficiently close to the unperturbed motion will tend asymptotically as $t \to \infty$ to one of the system SM, the corresponding strict local minimum of the function $U_0(x)$ for perturbed

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values of the constant integrals $U_i(x) = c_i$ $(i = 1, \ldots, k)$; in particular, the unperturbed motion is asymptotically stable for unperturbed values of the constant integrals used.

Theorem 3. If the SM does not make the function $U_0(x)$ even a non-strict minimum and is isolated for constant values of the integrals $U_i(x) = c_i$ $(i = 1, \ldots, k)$ of the motion along which the function $U_0(x)$ remains constant, then the unperturbed motion is unstable.

The theorems presented are a modification and further development of Routh's theorems /1—9/. Theorem 1 can be proved by an almost literal duplication of the Routh theorem given by Salvadori and Pozharitskii /5, 6/, and Theorems 2 and 3 by an almost literal duplication of the proofs of the Barbashin-Krasovskii theorem on asymptotic stability /10/ and the Krasovskii theorem on instability /11/, respectively. We note that Theorems 2 and 3 are analogues of the Rumyantsev Theorems VI and VII (/12/, pp.184-185).

It should be borne in mind that Theorems 1 and 2 assert the stability of the SM with respect to not all phase variables of the system under consideration, in general, but only with respect to those (or to their combinations) whose change increases the value of the function $U_0(x)$ for constant values of the integrals $U_i(x) = c_i$ $(i = 1, \ldots, k)$, i.e., with respect to part of the variables /12/.

We note that the assertion of Theorem 2 is analogous to some extent to the assertion of the Lyapunov-Malkin theorem /13/ on stability in the singular case of the critical case of several zero roots; however, unlike the latter, the application of Theorem 2 does not involve the compilation of a characteristic equation of the perturbed equations of motion and an analysis of its roots.

Theorems 2 and 3 are evidently applicable to a study of SM stability of not only scleronomic mechanical systems but also systems periodically dependent on time, while Theorem 1 is applicable for arbitrary rheonomic systems.

2. The equations of motion of a heavy inhomogeneous dynamically symmetric sphere along a horizontal plane with sliding friction (irrespective of the hypothesis about the nature of the friction) admit of /14, 15/ a function that does not grow along all motions of the sphere

$$U_0 = m(v_1^2 + v_2^2 + v_3^2) + J_1(\omega_1^2 + \omega_2^2) + J_3\omega_3^2 + 2mga\gamma_3 \leqslant 2h = \text{const} \tag{2.1}$$

and two integrals

$$U_1 = J_1(\omega_1\gamma_1 + \omega_2\gamma_2) + J_3\omega_3(\gamma_3 + a/\rho) = k = \text{const} \tag{2.2}$$

$$U_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{2.3}$$

Here m is the mass of the sphere J_1 and J_3 are the equatorial and axial central moments of inertia, respectively, g is the acceleration due to gravity, $-a$ is the coordinate of the geometric centre of the sphere on its dynamical axis of symmetry, measured from the centre of mass of the sphere (the positive direction of the axis is selected in such a way that $a > 0$), ρ is the radius of the sphere and v_i, ω_i and γ_i $(i = 1, 2, 3)$ are the respective components of the velocity vectors of the centre of mass of the sphere, its angular velocity, and the unit vector of the vertical in the principal central axes of inertia of the sphere.

According to Routh's theorem, critical points of the function (2.1) correspond to the CM of the sphere for constant values of the integrals (2.2) and (2.3), and we can reduce the problem of determining them to the problem of determining the critical points of the function

$$V = U_0 - 2\lambda(U_1 - k) + \mu(U_2 - 1) \tag{2.4}$$

where λ and μ are undetermined Lagrange multipliers. The function V takes stationary values if the variables v_i, ω_i, γ_i $(i = 1, 2, 3)$, λ and μ satisfy the system of equations (2.2), (2.3) and (2.4) $v_1 = v_2 = v_3 = 0$, $\omega_1 = \lambda\gamma_1$, $\omega_2 = \lambda\gamma_2$, $\omega_3 = \lambda(\gamma_3 + a/\rho)$ $(\mu - J_1\lambda^2)\gamma_1 = (\mu - J_1\lambda^2)\gamma_2 = 0$, $(\mu - J_3\lambda^2)\gamma_3 - J_3\lambda^2 a/\rho + mga = 0$.

This system has three groups of solutions (the relationships $v_1 = v_2 = v_3 = 0$ common for all these groups are omitted)

$$\omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0, \quad \omega_3 = \lambda(1 + a/\rho), \quad \gamma_3 = 1 \tag{2.5}$$
$$(\mu = J_3\lambda^2(1 + a/\rho) - mga)$$

$$\omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0, \quad \omega_3 = \lambda(-1 + a/\rho), \quad \gamma_3 = -1 \tag{2.6}$$
$$(\mu = J_3\lambda^2(1 - a/\rho) + mga)$$

$$\omega_1 = \lambda\gamma_1, \quad \omega_2 = \lambda\gamma_2, \quad \omega_3 = \lambda(\gamma_3 + a/\rho), \quad \gamma_1^2 + \gamma_2^2 = 1 - \gamma_3^2 \tag{2.7}$$
$$mga - J_3\lambda^2(\gamma_3 + a/\rho) + J_1\lambda^2\gamma_3 = 0 \quad (\mu = J_1\lambda^2)$$

The parameter λ is related to the constant k of the integral (2.2) by the respective relationships

$$k = J_3\lambda(1 + a/\rho)^2 \tag{2.5'}$$

$$k = J_3\lambda(-1 + a/\rho)^2 \tag{2.6'}$$

$$k = \frac{J_1[J_3(1 - a^2/\rho^2) - J_1]}{J_3 - J_1}\lambda + \frac{m^2g^2a^2}{(J_3 - J_1)\lambda^3} \tag{2.7'}$$

(it is assumed that $J_1 \neq J_3$).

Permanent rotations of the sphere around the vertically placed dynamical axis of symmetry correspond to the solutions (2.5) ((2.6)) when the centre of mass is above (below) its geometric centre; in particular for $\lambda = 0$ the equilibrium positons of the sphere correspond to the solutions of (2.5) and (2.6).

Regular precessions correspond to the solutions (2.7): the sphere rotates with angular velocity $\lambda a/\rho$ around the dynamical axis of symmetry precessing with angular velocity λ around the vertical passing through the centre of mass of the sphere and making a constant angle with the axis of symmetry, whose cosine equals

$$\gamma_3 = \alpha + \frac{mga}{(J_3 - J_1)\lambda^2}; \quad \alpha = \frac{J_3}{J_1 - J_3}\cdot\frac{a}{\rho} \tag{2.8}$$

The centre of mass of the sphere is fixed for all motions (2.5)−(2.7); the point of tangency of the sphere with the plane is fixed for the motions (2.5) and (2.6) and describes a circle on both the reference plane and on the surface of the sphere when the sphere performs pure roll, in the motions (2.7).

Regular precessions of the sphere obviously do not exist for all values of the parameter λ but only for those for which $|\gamma_3| < 1$ follows from (2.8).

Finally, we note that taking (2.5)−(2.7) into account follows from (2.5')−(2.7') that for each value of k there exists not more than four SM of the sphere of the form (2.5)−(2.7): two permanent rotations ("upper" and "lower") and not more than two regular precessions (it follows from (2.7') that not more than two different real values of λ can correspond to one value of k). All (four in the general case) SM of the sphere are isolated for fixed values of k from each other for γ_3 determinable from (2.8) and not equal to ± 1, i.e., for

$$\lambda^2/\lambda_1^2 \neq 1, \ \lambda^2/\lambda_2^2 \neq 1 \tag{2.9}$$

$$\lambda_1^2 = \frac{mga}{J_3(1 + a/\rho) - J_1}, \quad \lambda_2^2 = \frac{mga}{J_1 - J_3(1 - a/\rho)}$$

3. When the condition /14, 15/

$$\lambda^2/\lambda_1^2 > 1 \tag{3.1}$$

is satisfied the SM (2.5) makes the function (2.1) a strict minimum for constant values of the integrals (2.2) and (2.3) and therefore stable according to Theorem 1 with respect to the variables v_i, ω_i, γ_i $(i = 1, 2, 3)$. Under the condition

$$\lambda^2/\lambda_1^2 < 1 \tag{3.2}$$

the SM (2.5) does not make the function U_0 even a non-strict minimum.

Analogously, when the condition

$$\lambda^2/\lambda_2^2 < 1 \tag{3.3}$$

is satisfied the SM (2.6) makes the function (2.1) a strict minimum and therefore stable, while when

$$\lambda^2/\lambda_2^2 > 1 \tag{3.4}$$

it does not make the function U_0 even a non-strict minimum.

Finally, when

$$J_1 > J_3 \tag{3.5}$$

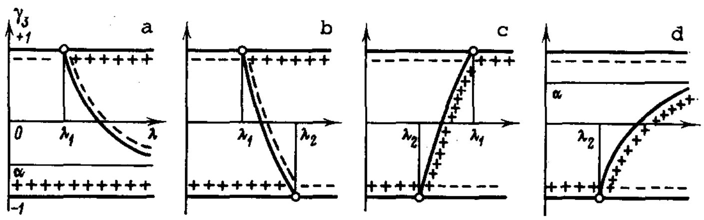
the SM (2.7) makes the function (2.1) a strict minimum for constant values of the integrals (2.2) and (2.3), and therefore, stable according to Theorem 1 with respect to the variables v_i $(i = 1, 2, 3)$, $\omega_j - \lambda\gamma_j$ $(j = 1, 2)$, ω_3, γ_3, while when

$$J_1 < J_3 \tag{3.6}$$

it does not make the function U_0 even a non-strict minimum.

It will be shown below that the function U_0 remains constant only on a SM of the form (2.5)−(2.7). Since these SM are isolated from each other when conditions (3.1)−(3.4) are satisfied (see (2.9)), according to Theorems 2 and 3 the permanent rotations (2.5) and (2.6) are stable, where it is asymptotic with respect to the variables v_i, γ_i $(i = 1, 2, 3)$ and ω_j $(j = 1, 2)$ when conditions (3.1) and (3.3) are satisfied, respectively, and unstable when conditions (3.2) and (3.4) are satisfied, respectively. Analogously, the regular precessions (2.7) are stable, where the stability is asymptotic with respect to the variables v_i $(i = 1, 2, 3)$, $\omega_j - \lambda\gamma_j$ $(j = 1, 2)$ and $mga - J_3\lambda\omega_3 + J_1\lambda^2\gamma_3$, when condition (3.5) is satisfied and unstable when (3.6) is satisfied.

In other words, if the centre of mass of the sphere is above its geometric centre in the permanent rotations, they are stable when $J_3(\rho + a) - J_1\rho > 0$ $(\lambda_1^2 > 0)$ and the angular velocity of the sphere is sufficiently large $(\omega_3^2 > \omega_*^2 = \lambda_1^2(1 + a/\rho)^2)$, and unstable otherwise; if the centre of mass of the sphere is below its geometric centre during the permanent rotations, then they are always stable when $J_1\rho - J_3(\rho - a) \leqslant 0$ $(\lambda_2^2 \leqslant 0)$, and for $J_1\rho - J_3(\rho - a) > 0$ when the angular velocity is sufficiently small $(\omega_3^2 < \omega_{**}^2 = \lambda_2^2(1 + a/\rho)^2)$, and unstable otherwise; regular precessions of the sphere are stable (unstable) irrespective of the magnitudes of the precession and intrinsic rotation angular velocities if the axial moment of inertia of the sphere is smaller (larger) than the equatorial value.



The set of SM (2.5)—(2.7) is represented geometrically in the half-plane $\lambda \geqslant 0$, γ_3 (see the figure) (for $\lambda < 0$ the bifurcation diagram is obtained from the mentioned specular reflection in the γ_3 axis). The rectilinear and curvilinear branches of the SM curve, respectively, correspond to permanent rotations (2.5) and (2.6) and to regular precessions (2.7). The distribution of stable SM (marked with the plus symbol) and unstable SM (minus) is subject to the laws of change in stability for a fixed value of the parameter λ; the change in stability occurs only at the bifurcation points. The parts a—d of the figure correspond, respectively, to the cases

a) $J_3(1 - a/\rho) \geqslant J_1$, b) $J_3 > J_1 > J_3(1 - a/\rho)$.
c) $J_3(1 + a/\rho) > J_1 > J_3$, d) $J_3(1 + a/\rho) \leqslant J_1$.

We note that in cases b) and c), as for a "tip-top", top the effect of losses in stability (hard and soft, respectively) in the rotation of a sphere with the lowest location of the mass is observed as the angular velocity increases that accompanies stabilization of the rotation of a sphere with the highest position of the centre of mass.

Remark. In the case of viscous sliding friction, the characteristic equation of the perturbed equations of motion of a sphere in the neighbourhood of each of the SM (2.5)—(2.7) has the form $p^3 f(p) = 0$. When conditions (3.2), (3.4), (3.6) are satisfied at least one root of the equation $f(p) = 0$ lies in the right half-plane and the corresponding motions are unstable. When conditions (3.1), (3.3) and (3.5) are satisfied, all the roots of the equation $f(p) = 0$ lie in the left half-plane and since all the conditions of the Lyapunov-Malkin theorem are satisfied, the corresponding motions are stable, where the stability is asymptotic in parts of the variables /15, 18/.

We note that analogous results are obtained in this paper for any law of sliding friction.

4. We prove that the function U_0 in (2.1) actually retains constant values only on SM of the form (2.5)—(2.7).

The function U_0 obviously does not decrease if and only if the sphere rolls on the plane without sliding. The relationships

$$v_1 = (\rho\gamma_3 + a)\,\omega_2 - \rho\gamma_2\omega_3,\quad v_2 = -(\rho\gamma_3 + a)\,\omega_1 + \rho\gamma_1\omega_3,\quad v_3 = \rho\,(\omega_1\gamma_2 - \omega_2\gamma_1) \tag{4.1}$$

that express the vanishing of the slip velocity of the sphere must be satisfied here, while the equations of motion of the sphere can be represented in the form (N is the magnitude of the normal reaction of the reference plane)

$$\begin{aligned} &v_1^{\cdot} + \omega_2 v_3 - \omega_3 v_2 = (Nm^{-1} - g)\,\gamma_1,\quad v_2^{\cdot} + \omega_3 v_1 - \omega_1 v_3 = \\ &\quad (Nm^{-1} - g)\,\gamma_2,\quad v_3^{\cdot} + \omega_1 v_2 - w_2 v_1 = (Nm^{-1} - g)\,\gamma_3 \end{aligned} \tag{4.2}$$

$$\begin{aligned} &J_1\omega_1^{\cdot} + (J_3 - J_1)\,\omega_2\omega_3 = Na\gamma_2,\quad J_1\omega_2^{\cdot} + (J_1 - J_3)\,\omega_3\omega_1 = \\ &\quad -Na\gamma_1,\quad J_3\omega_3^{\cdot} = 0 \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\gamma_1^{\cdot} + \omega_2\gamma_3 - \omega_3\gamma_2 = 0,\quad \gamma_2^{\cdot} + \omega_3\gamma_1 - \omega_1\gamma_3 = 0 \\ &\gamma_3^{\cdot} + \omega_1\gamma_2^{\cdot} - \omega_2\gamma_1 = 0. \end{aligned} \tag{4.4}$$

Moreover, the relationships (2.2), (2.3) and

$$J_1(\omega_1^2 + \omega_2^2) + J_3\omega_3^2 + 2mga\gamma_3 = 2h = \mathrm{const} \tag{4.5}$$

are satisfied.

Differentiating the first two relationships (4.1) with respect to time and eliminating $\omega_i^{\cdot}$ and $\gamma_i^{\cdot}$ $(i = 1, 2, 3)$ using (4.3) and (4.4), we substitute the expressions obtained and the relationships (4.1) into the first two equations in (4.2)

$$Nm^{-1}A\gamma_j = J_1 g\gamma_j + B\rho\omega_3\omega_j \quad (j = 1, 2) \tag{4.6}$$
$$A = J_1 + ma(\rho\gamma_3 + a), \quad B = J_3(\gamma_3 + a/\rho) - J_1\gamma_3$$

Multiplying the first relationship in (4.6) by γ_2, the second by $-\gamma_1$, and adding term by term, we obtain $B\rho\omega_3(\omega_1\gamma_2 - \omega_2\gamma_1) = 0$ (if $\gamma_1 = \gamma_2 = 0$, then the motions satisfying (4.5) are only the permanent rotations).

It hence follows that either $\omega_3 = 0$ or

$$\omega_1\gamma_2 - \omega_2\gamma_1 = 0 \tag{4.7}$$

(if $B = 0$, then $\gamma_3 = \alpha = \text{const}$ and (4.7) again follows from (4.4)).

We assume first that $\omega_3 = 0$. Then we obtain the relationships

$$A\omega_1^{\cdot} = mga\gamma_2, \quad A\omega_2^{\cdot} = -mga\gamma_1 \tag{4.8}$$

from (4.6) and (4.3), and differentiating the identity (4.5) with respect to time and taking account of (4.8), we obtain the relationship

$$A\gamma_3^{\cdot} = J_1(\omega_2\gamma_1 - \omega_1\gamma_2) \tag{4.9}$$

Comparing (4.9) and (4.4), we conclude that $\omega_2\gamma_1 - \omega_1\gamma_2 = 0$, i.e., pure roll of the sphere is possible only when condition (4.7) is satisfied.

The relationships

$$\omega_1 = \lambda\gamma_1, \quad \omega_2 = \lambda\gamma_2, \quad \omega_3 = \text{const}, \quad \gamma_3 = \text{const}, \quad v_3 = 0, \tag{4.10}$$
$$v_1 = w\gamma_2, \quad v_2 = -w\gamma_1; \quad w = (\rho\gamma_3 + a)\lambda - \rho\omega_3$$

must here be satisfied (see (4.1) and (4.3)), where $\lambda = \text{const}$, as follows from (2.2). Substituting (4.10) into (4.2), we obtain

$$(N - mg)\gamma_i = mw\lambda\gamma_3\gamma_i \quad (i = 1, 2)$$
$$(N - mg)\gamma_3 = -mw\lambda(1 - \gamma_3^2).$$

Multiplying the last relationships by $\gamma_1, \gamma_2, \gamma_3$, respectively, and adding term by term, we obtain $N = mg$, i.e.

$$[(\rho\gamma_3 + a)\lambda - \rho\omega_3]\lambda(1 - \gamma_3^2) = 0$$

from which it follows that pure roll of the sphere is possible only for SM of the form (2.5)–(2.7).

We note that an analogous deduction also follows from the results in /19/, devoted to a description of the limit values of a heavy solid on a plane with viscous sliding friction.

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STABILITY UNDER CONSTANTLY ACTING PERTURBATIONS, AND AVERAGING IN AN UNBOUNDED INTERVAL IN SYSTEMS WITH IMPULSES*

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The question of the closeness of non-stationary solutions of the exact and averaged equations in an unlimited time interval is investigated for ordinary differential equations whose right sides contain generalized functions of time (generalized derivatives of functions of bounded variation). The appropriate assertions in the development of the method proposed in /1/ are derived from a special theorem on stability under permanently acting perturbations. The results obtained (more general in the case of equations with smooth coefficients then the assertions in /2, 3/) afford an apportunity for giving a foundation to the applicability of the averaging method to quasiconservative vibration impact systems /4/.

We note that the question of the correspondence between solutions of the exact equations and the stationary solutions of the average equations was investigated in /5/ (see /6/ also) for systems in standard form with impulsive action.

1. We shall use the following notation: R^n is a Euclidean n-space, $|x|$ is the norm of the element $x \in R^n$, I is the interval $[0, \infty)$, $B_x(K) = \{x: x \in R^n, |x| \leqslant K\}$, $G = I \times B_x(K)$. We shall henceforth consider integrals of the form

$$\int_{t_1}^{t_2} f(s, x(s))\,du(s), \quad (t_1, t_2) \in J \tag{1.1}$$

which are understood to be Lebesgue-Stieltjes integrals. We shall say with respect to the integrating function $u(t)$ that $u(t) \in BU(J)$ if $u(t)$ is a scalar function defined for $t \in J$ and possessing the following properties:

1) $u(t)$ is continuous on the right and is of limited variation in each compact subinterval of the interval J;

2) The discontinuities $t_1 < t_2 < \ldots$ $(t_1 \geqslant t_0 \geqslant 0)$ of the function $u(t)$ have the single limit point $+\infty$.

Functions defined on J with values in $B_x(K)$ continuous to the right and with the same points of discontinuity of the first kind as $u(t)$ will be considered as $x(t)$. Then if $f(t, x)$ is a function defined in G with values in R^n bounded in the norm, continuous in x uniformly with respect to t and having not more than a denumerable number of points of discontinuity of the first kind in t, the integral (1.1) exists. We note that with the above assumptions, the appropriate generalization of the Riemann-Stieltjes integral can be used in place of the Lebesgue-Stieltjes integral. Later, if the question of the existence of the integral (1.1) is not especially stipulated, we shall assume that the listed conditions are satisfied.

For the function $f(t, x)$ defined in G and integrable with respect to $u(t) \in BU(J)$, we introduce

$$S_x(f) = \sup_{|t_2 - t_1| \leqslant 1} \left| \int_{t_1}^{t_2} f(s, x)\,du(s) \right|, \quad x \in B_x(K)$$

Lemma 1. Let the function $f(t, x)$ be defined on G and continuous in x uniformly with respect to $t \in J$. Let the function $x(t)$ which is continuous to the right with values in

*Prikl.Matem.Mekhan., 50, 1, 50-56, 1986